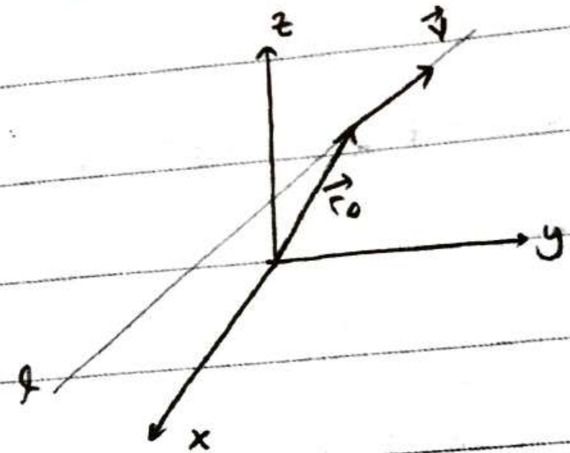
12.5 LINES and PLANES in SPACE



$$\ell : \underbrace{\vec{r}(t) = \vec{r_0} + t\vec{v}}$$

vector equation

$$\vec{r_0} = x_0 i + y_0 j + z_0 k$$

$$\vec{v} = ai + bj + ck$$

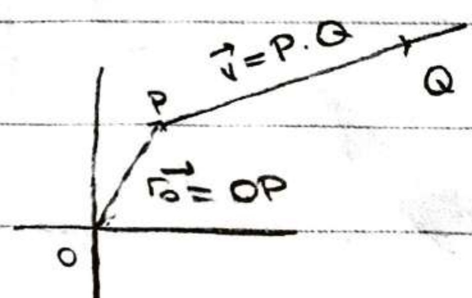
$$\vec{r}(t) = x(t) i + y(t) j + z(t) k$$

parametric equations
$$\begin{cases} x(t) = x_0 + at \\ y(t) = y_0 + bt \\ z(t) = z_0 + ct \end{cases}, \quad t \in \mathbb{R}$$

if $a \neq 0$, $b \neq 0$, $c \neq 0$

standart equation for the line.
$$\left\{ \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} \right.$$

ex: Find parametric equations for the line that passes from $P(1,0,2)$ $Q(-1,3,0)$



$$\vec{r_0} = i + 2k$$

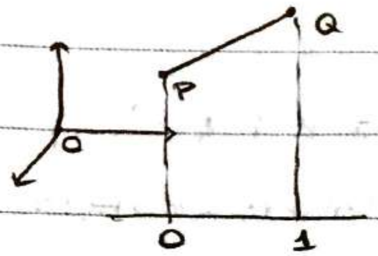
$$\vec{v} = \vec{PQ} = -2i + 3j - 2k$$

vectoral equations $\leftarrow$ $\{ \vec{r}(t) = i + 2k + t(-2i + 3j - 2k)$

parametric equations
$$\begin{cases} x = 1 + 2t \\ y = 3t \\ z = 2 - 2t \end{cases}$$

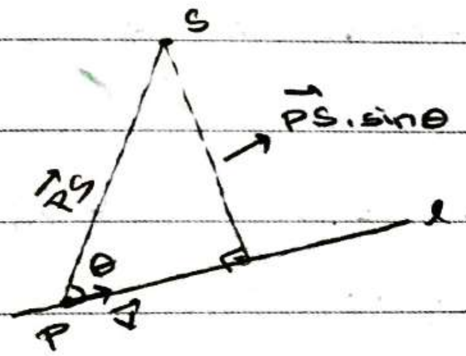
standart equations
$$\left\{ \frac{x-1}{-2} = \frac{y}{3} = \frac{z-2}{-2} \right.$$

Line segment between P, Q



$r(t) = \vec{OP} + t\,\vec{PQ} \quad 0 \le t \le 1$

The distance from point S to Line Through P parallel to $\vec{v}$

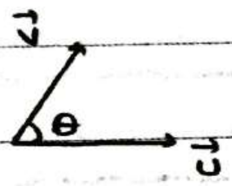


$$\text{distance} = |\vec{PS}| \cdot \sin\theta$$

$$= \frac{|\vec{PS}| \cdot |\vec{v}| \sin\theta}{|\vec{v}|}$$

$$\text{distance} = \frac{|\vec{PS} \times \vec{v}|}{|\vec{v}|}$$

recall:



$$|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \cdot \sin\theta$$

ex: Find the distance from the point $S(1,1,5)$ to the line

$L: \; x = 1+t \; , \; y = 3-t \; , \; z = 2t$

$\vec{v} = i - j + 2k$ direction vector of L. Choose a point on L: $t = 0 \Rightarrow P(1,3,0)$

$$\vec{PS} = (1-1)i + (1-3)j + (5-0)k$$

$$= -2j + 5k$$

$$\text{distance} = \frac{|\vec{PS} \times \vec{v}|}{|\vec{v}|}$$

$$\vec{PS} \times \vec{v} = \begin{vmatrix} i & j & k \\ 0 & -2 & 5 \\ 1 & -1 & 2 \end{vmatrix} = i(-2.2-(-1).5) - j(0.2-5.1) + k(0.(-1)-(-2)1)$$

$$= i + 5j + 2k$$

$$|\vec{PS} \times \vec{v}| = \sqrt{1^2+5^2+2^2} = \sqrt{30}$$

$$|\vec{v}| = \sqrt{1^2+1^2+2^2} = \sqrt{6}$$

$$\text{distance} = \frac{\sqrt{30}}{\sqrt{6}} = \sqrt{5}$$

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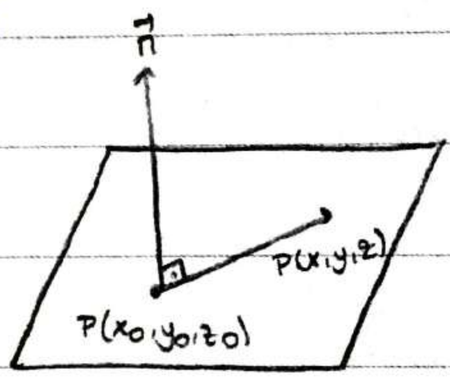


$$\vec{P_0P} \cdot \vec{n} = 0$$

$$\vec{n} = a\hat{i} + b\hat{j} + ck$$

$$\vec{P_0P} = (x-x_0)\,\hat{i} + (y-y_0)\hat{j} + (z-z_0)k$$

$$\vec{P_0P} \cdot \vec{n} = a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$$

an equation for the plane.

$ax+by+cz=d : ax_0+by_0+cz_0$

ex: The normal vector of the plane

$3x-2y+5z=24$

is

$\vec{n} = 3i - 2j + 5k$

ex: Find an equation for the plane that passes through the point $P(1,3,5)$ and has normal

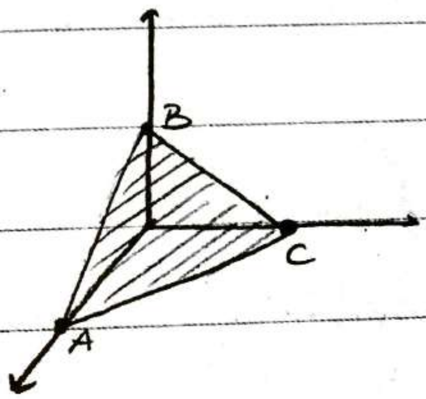
$\vec{n} = i - 2k$

answer:

$1(x-1) + 0(y-3) - 2(z-5) = 0$

$x - 2z = 11$

ex: Find an equation for the plane through $A(1,0,0)$ $B(0,0,2)$ $C(0,3,0)$



$\vec{n} = \vec{AC} \times \vec{AB}$

$$= \begin{vmatrix} i & j & k \\ -1 & 3 & 0 \\ -1 & 0 & 2 \end{vmatrix}$$

$= i(6-0) - j(-2-0) + k(0+3)$

$= 6i + 2j + 3k$

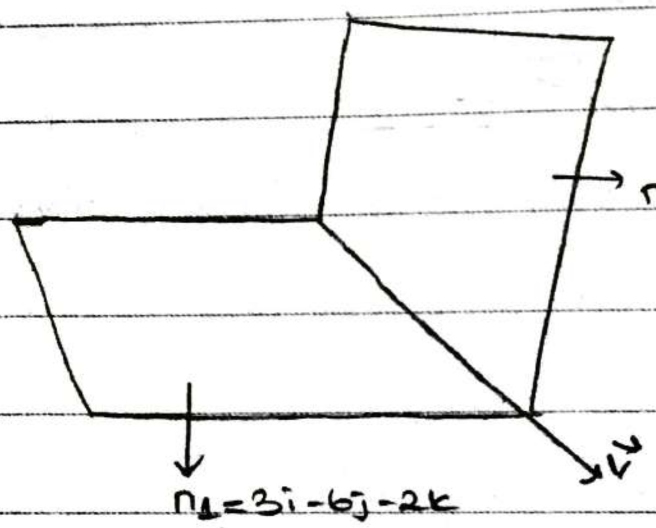
The plane passing from $A(1,0,0)$ and has normal $6i + 2j + k$ has an equation

$6(x-1) + 2(y-0) + 3(z-0) = 0$

Lines of Intersection

ex: Find parametric equations for the line which is the intersection of the planes

$$3x-6y-2z=15$$
$$2x+y-2z=5$$



$$\vec{v} = \vec{n_1} \times \vec{n_2} = \begin{vmatrix} i & j & k \\ 3 & -6 & -2 \\ 2 & 1 & -2 \end{vmatrix} = 14i+2j+15k$$

$z=0 \Rightarrow 3x-6y=15$

$2x+y=5$ 　 $15x=45$

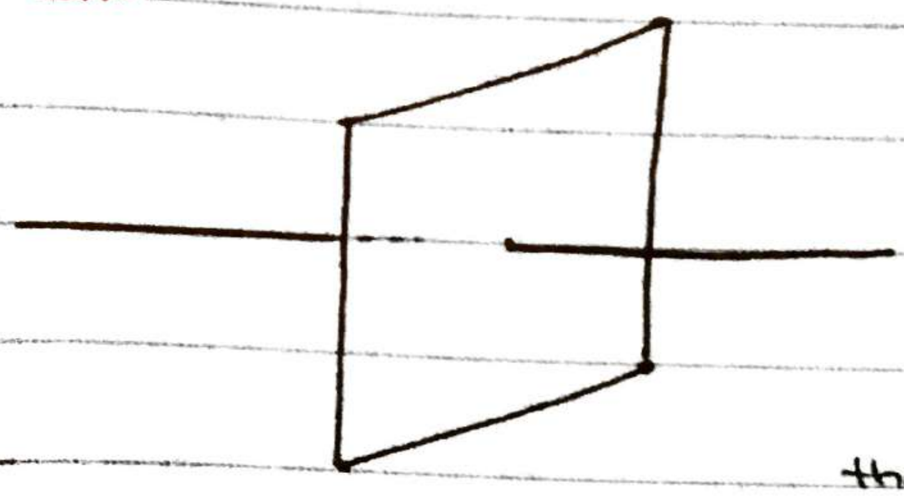
$6x+6y=30$ 　 $x=3$ 　 $y=-1$

$(3,-1,0)$ is a point that lies on the intersection.

The line passing from $P(3,-1,0)$ and having direction $\vec{v}=14i+2j+15k$ has an equation.

$$\boxed{\begin{aligned} x &= 3+14t \\ y &= -1+2k \\ z &= 15t \end{aligned}}, \quad t \in \mathbb{R}$$

ex:



Find the point where the line $x = \frac{8}{3} + 2t$

$y = -2t$, $z = 1 + t$ intersects

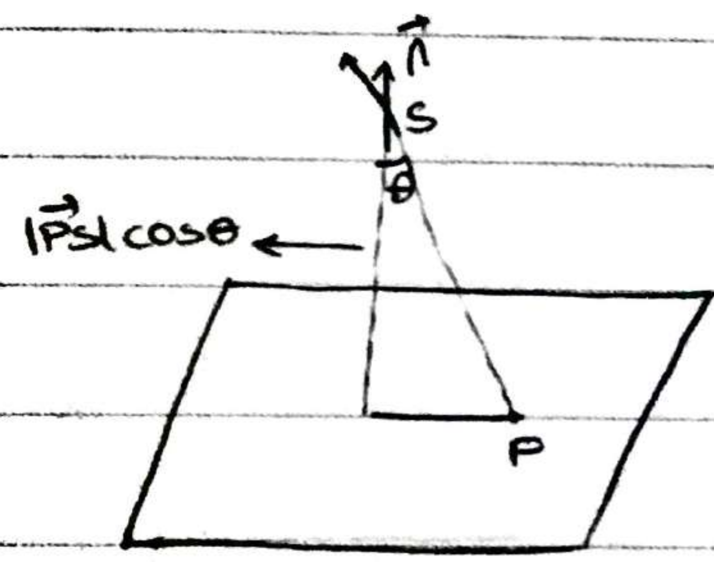
the plane $3x + 2y + 6z = 6$

$$3\left(\frac{8}{3} + 2t\right) + 2(-2t) + 6(1+t) = 6$$

$$t = -1$$

$$P(2/3, 2, 0)$$

Distance From Point to a Plane



$$\text{distance} = \left||PS| \cdot \cos\theta\right|$$

$$= \frac{|PS|\,|\vec{n}| \cdot \cos\theta}{|\vec{n}|}$$

$$= \frac{|\vec{PS} \cdot \vec{n}|}{|\vec{n}|}$$

ex: Find the distance from $S(1, 1, 3)$ to the plane

$3x + 2y + 6z = 6$

$P(0, 0, 1)$ is on the plane

$\vec{n} = 3i + 2j + 6k$

$\vec{PS} = i + j + 2k$

$$\text{distance} = \frac{|\vec{PS} \cdot \vec{n}|}{|\vec{n}|} = \frac{|3.1 + 2.1 + 6.2|}{\sqrt{3^2 + 2^2 + 6^2}} = \frac{17}{7}$$

ex: Find the angle between the planes $3x-6y-2z=15$ and $2x+y-2z=5$

$$\vec{n_1} = 3i-6j-2k$$

$$\vec{n_2} = 2i+j-2k$$

$$\vec{n_1}\cdot\vec{n_2} = |\vec{n_1}||\vec{n_2}|\cos\theta$$

$$\cos\theta = \frac{3.2+(-6).1+(-2)(-2)}{\sqrt{3^2+6^2+2^2}\,\sqrt{2^2+1^2+2^2}} = \frac{4}{21}$$

$$\theta = \cos^{-1}\left(\frac{4}{21}\right) \approx 1.38 \text{ radians} \approx 79°$$

Ex: (10.1 / 77)

$$\lim_{n\to\infty}\left(\frac{n^2}{2n-1}\sin\frac{1}{n}\right) = ?$$

$$= \lim_{n\to\infty}\frac{n}{2n-1}\cdot n\cdot\sin\frac{1}{n}$$

$$\Rightarrow \lim_{n\to\infty}\frac{n}{2n-1} = \frac{1}{2}$$

$$\Rightarrow \lim_{n\to\infty} n\sin\frac{1}{n} = \lim_{n\to\infty}\frac{\sin\frac{1}{n}}{\frac{1}{n}} = 1$$

$$\rightarrow \lim_{n\to\infty}\frac{n}{2n-1}\cdot n\cdot\sin\frac{1}{n} = \frac{1}{2}\cdot 1 = \frac{1}{2}$$ (converge)

Ex: (10.2 / 46)

$$\sum_{n=1}^{\infty} \frac{1}{2^{1/n}} - \frac{1}{2^{1/n+1}} \quad \text{find its sum?}$$

$$S_N = \left(\frac{1}{2} - \cancel{\frac{1}{2^{1/2}}}\right) + \left(\cancel{\frac{1}{2^{1/2}}} - \cancel{\frac{1}{2^{1/3}}}\right) + \dots + \left(\cancel{\frac{1}{2^{1/N}}} - \frac{1}{2^{1/N+1}}\right)$$

$$= \frac{1}{2} - \frac{1}{2^{1/N+1}}$$

$$\sum_{n=1}^{\infty} \frac{1}{2} - \frac{1}{2^{1/N+1}} = \lim_{n\to\infty} S_N = \frac{1}{2} - \frac{1}{1} = -\frac{1}{2} \quad \text{(converge)}$$

Ex: (10.2 / 68)

$$\sum_{n=0}^{\infty} \frac{e^{n\pi}}{\pi^{ne}} \quad \text{converges or diverges?}$$

$$= \sum_{n=0}^{\infty} \left(\frac{e^{\pi}}{\pi^{e}}\right)^{n}$$

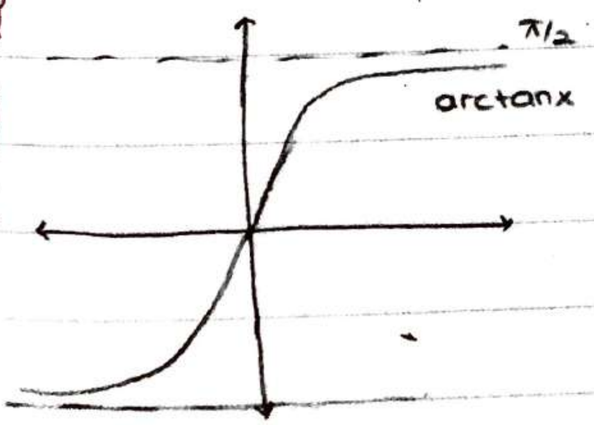
$$\downarrow$$

$$\frac{2.71^{3.14}}{3.14^{2.71}} = \frac{23.14}{22.46} > 1 \Rightarrow \text{diverges}$$

Ex: (10.3 / 37)

$$\sum_{n=1}^{\infty} \frac{8 + \tan^{-1} n}{1 + n^2} \quad \text{converges or diverges?}$$

Comparison Test



$n \geq 1 \quad 0 \leq \arctan n \leq \pi/2$

$$0 \leq \sum_{n=1}^{\infty} \frac{8 + \tan^{-1} n}{1 + n^2} \leq \underbrace{\sum_{n=1}^{\infty} \frac{8\ \pi/2}{1 + n^2}}$$

$$\downarrow$$

$$4\pi \sum_{n=1}^{\infty} \frac{1}{1+n^2}$$

(p=2 series)

(converges)

* **Ex:** (10.3 / 37)

$$\sum_{n=1}^{\infty} \frac{8\tan^{-1} n}{1+n^2} \quad \text{converges or diverges?}$$

Integral Test

$$\int_1^{\infty} \frac{8\tan^{-1} x}{1+x^2}\,dx = \int_{\pi/4}^{\pi/2} 8u\,du = 4u^2 \Big|_{\pi/4}^{\pi/2} < +\infty$$

$\tan^{-1} x = u$

$\frac{dx}{1+x^2} = du$

Integral converges $\Rightarrow$ series converges

* **Ex:** (10.5 / 36)

$$\sum_{n=1}^{\infty} \frac{n\,2^n\,(n+1)!}{3^n \cdot n!} \quad \text{converges or diverges?}$$

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\dfrac{(n+1)\,2^{n+1}\,(n+2)!}{3^{n+1}\cdot(n+1)!}}{\dfrac{n\cdot 2^n\,(n+1)!}{3^n\cdot n!}}$$

$$= \lim_{n\to\infty} \frac{\cancel{(n+1)}\cdot 2\,(n+2)}{n\cdot 3\cdot\cancel{(n+1)}}$$

$$= \frac{2}{3}\lim_{n\to\infty} \frac{n+2}{n} = \frac{2}{3} < 1 \Rightarrow \text{series converges by ratio test}$$

* **Ex:** (10.5 / 40) (root test)

$$\sum_{n=2}^{\infty} \frac{n}{(\ln n)^{n/2}}$$ converges or diverges ?

$$\lim_{n\to\infty} \left(\frac{n}{(\ln n)^{n/2}}\right)^{1/n} = \lim_{n\to\infty} \frac{n^{1/n}}{(\ln n)^{1/2}} = \frac{1}{\infty} = 0 < 1$$

→ By root test , the series converges.

* **Ex:** (10.6 / 25)

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1+n}{n^2}$$ converges absolutely , converges conditionally or diverges ?

a) converges absolutely ?

$$\sum_{n=1}^{\infty} \left| (-1)^{n+1} \frac{1+n}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1+n}{n^2} \geq \sum_{n=1}^{\infty} \frac{n}{n^2} = +\infty \quad \left(\text{harmonic series}\right)$$

→ So series diverge absolutely.

b) converges conditionally ?

i) $a_n = \frac{1+n}{n^2} \geq 0$

ii) $a_n = \frac{1}{n^2} + \frac{1}{n}$ $\quad a_{n+1} \leq a_n$ decreasing

iii) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1+n}{n^2} = 0$

→ So series converge by Alternating Series Test conditionally

* Ex: (10.7 / 18)

Find the series radius and interval of converg...

$$\sum_{n=0}^{\infty} \frac{n x^n}{4^n (n^2+1)}$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{\dfrac{(n+1)\, x^{n+1}}{4^{n+1}\left((n+1)^2+1\right)}}{\dfrac{n x^n}{4^n (n^2+1)}}$$

$$= \lim_{n\to\infty} \left| \frac{(n+1)x}{4\left((n+1)^2+1\right)} \right| \frac{(n^2+1)}{n}$$

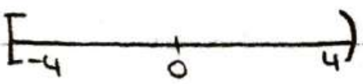
$$= \lim_{n\to\infty} \frac{n+1}{n} \cdot \frac{n^2+1}{n^2+2n+2} \left| \frac{x}{4} \right|$$

$$= 1 \cdot 1 \left| \frac{x}{4} \right|$$

if $\left| \frac{x}{4} \right| < 1 \Rightarrow$ series converges absolutely

$\quad > 1 \Rightarrow$ series diverges

Radius of converge $= 4$



$x = 4 \Rightarrow \sum_{n=0}^{\infty} \frac{n}{n^2+1} = +\infty$

$x = -4 \Rightarrow \sum_{n=0}^{\infty} (-1)^n \cdot \frac{n}{n^2+1}$ converge conditionally by AST

Interval of convergence $[-4, 4)$

series converge absolutely for $|x| < 4$

Series converge conditionally for $x = -4$

Series divergence elsewhere

* **Ex:** (10.7 / 50)

Find the Mclaurin series of

$$f(x) = \frac{3}{x-2}$$

$$\left\{ \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \ , \quad |x| < 1 \right.$$

$$\frac{3}{x-2} = \frac{3}{-2} \frac{1}{1-\frac{x}{2}} = \frac{3}{-2} \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n , \quad \left|\frac{x}{2}\right| < 1$$

$$\frac{3}{x-2} = -\frac{3}{2} \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n , \quad |x| < 2$$

* **Ex:** (10.8 | 1)

Find the Taylor Series of

$f(x) = e^{2x}$ about $x = 0$

$f(0) = e^0 = 1$

$f'(x) = 2e^{2x}$

$f''(x) = 4e^{2x}$

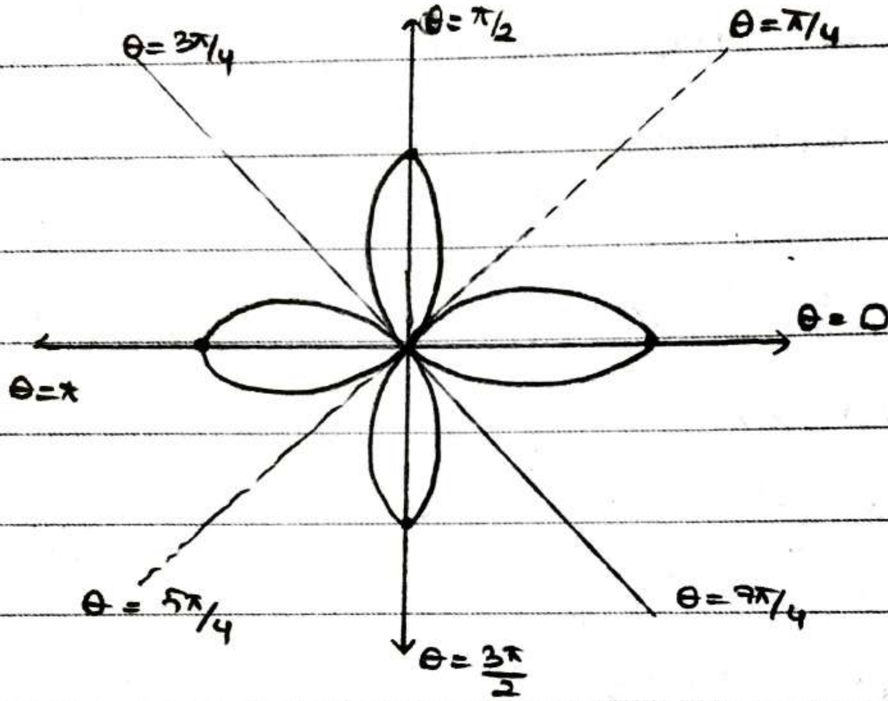
$f^{(n)}(0) = 2^n$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} (x-0)^n$$

$$e^{2x} = \sum_{n=0}^{\infty} \frac{2^n}{n!} x^n \quad \text{for all } x$$

* **Ex:** (11.4 / 20)

Graph $r = \cos 2\theta$



$\theta = \pi \rightarrow r = 1$

$\theta = \pi/2 \rightarrow r = -1$